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ON THE VAN DER POL RELAXATION OSCILLATOR WITH A SINUSOIDAL FORCING TERM

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ABSTRACT

Asymptotic approximations of subharmonic solutions of the periodically forced Van der Pol relaxation oscillator are constructed with singular perturbation techniques. These approximations are locally valid and may take the form of a two variable expansion in one region and a boundary layer type of solution in a next region. Integration constants are determined by averaging and matching conditions. The construction of the approximations brings about certain restricting conditions on the amplitude of the forcing term.

KEY WORDS & PHRASES: *Van der Pol equation, relaxation oscillation, subharmonic entrainment, singular perturbation*

1. INTRODUCTION

In this paper we study the Van der Pol equation with a sinusoidal forcing term

$$(1.1)\qquad \frac{d^2x}{dt^2} + \nu(x^2-1)\frac{dx}{dt} + x = (\alpha\nu+\beta)\cos t,$$

for large values of the parameter ν and with $0 < \alpha < 2/3$. Using singular perturbation techniques we construct a formal asymptotic approximation of the $2\pi(2n-1)$-periodic solution with $n = O(\nu)$. In the process of constructing such approximation we arrive upon a set of conditions for α, β and ν. These conditions are such that for a given α, the parameter β lies on an interval

$$(1.2)\qquad \underline{\beta}_n(\alpha) < \beta < \bar{\beta}_n(\alpha).$$

These intervals overlap, so that for β on the interval $(\underline{\beta}_n(\alpha), \bar{\beta}_{n+1}(\alpha))$ two solutions with period $T = 2\pi(2n\pm1)$ may coexist. In earlier studies [3,4 and 5] asymptotic solutions for the cases $\alpha = 0$ and $\alpha = 2/3$ have been constructed. In our analysis of the present problem we see that in the asymptotic solution elements of both cases can be distinguished. A periodic solution of (1.1) has a behaviour that is characteristic for singularly perturbed type of problems. Locally the solution has a boundary layer type behaviour like one meets in problems of fluid mechanics. On the other hand the solution also passes a large time interval, where a two time-scales expansion can be made. Finally, we distinguish a sequence of points, determined by the intersections with the lines $x = \pm1$, where the local behaviour of the solution is analyzed by a stretching procedure in both the dependent and independent variable. For a complete picture of the different regions which are successively passed through by the solution we refer to Fig. 1.1. Integration constants in locally valid asymptotic solutions are determined by averaging conditions and by matching pairs of local solutions of adjacent regions.

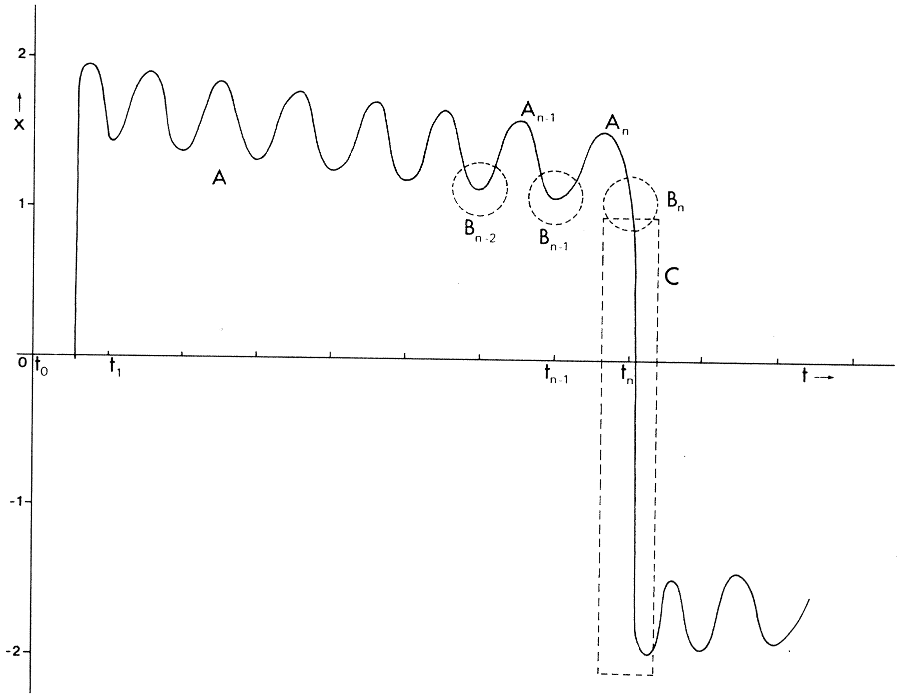

Fig. 1.1. Regions with a local asymptotic solution

2. ASYMPTOTIC SOLUTION FOR REGION A

In this region the solution exhibits an oscillatory behaviour of period 2π in the regular time scale and, at the same time, a slow decrease of its average value. In order to analyze the solution asymptotically we introduce an additional time variable $\tau = (t-t_0-\pi)/\nu$ with $t_0 = \pi/2 \pmod{2\pi}$ and consider the following two-variable expansion for the solution (see COLE [1]).

$$x = x_0(t,\tau) + \nu^{-1}x_1(t,\tau) + \nu^{-2}x_2(t,\tau) + \dots \tag{2.1}$$

Substituting (2.1) into (1.1) and equating the terms with equal powers of ν we obtain a recurrent system of differential equations for $x_i(t,\tau)$. The first equation reads

$$(x_0^2-1)\frac{\partial x_0}{\partial t} = \alpha \cos t \tag{2.2}$$

or

(2.3) $$\frac{1}{3}x_0^3 - x_0 = \alpha\ \mathrm{sin}t + C_0(\tau)$$

having a solution of the form

(2.4) $$x_0 = 2\cos[\frac{1}{3}\arccos\{\frac{3}{2}\alpha\ \mathrm{sin}t + \frac{3}{2}C_0(\tau)\}].$$

The second equation of the iterative scheme becomes

(2.5) $$\frac{\partial^2 x_0}{\partial t^2} + (x_0^2-1)(\frac{\partial x_0}{\partial \tau}+\frac{\partial x_1}{\partial t}) + 2x_0x_1\frac{\partial x_0}{\partial t} + x_0 = \beta\ \mathrm{cos}t$$

or, after integration with respect to t and with the use of (2.3),

(2.6) $$(x_0^2-1)x_1 = -\frac{\partial x_0}{\partial t} - \int_{t_0+\pi}^{t} G_0(\bar{t},\tau)d\bar{t} + \beta\ \mathrm{sin}t + C_1(\tau)$$

with

(2.7) $$G_0(t,\tau) = x_0(t,\tau) + \partial C_0/\partial\tau.$$

The integral of G_0 in the right-hand side of equation (2.6) is secular in the sense that for $(t-t_0-\pi) \to \infty$ this term would increase in order of magnitude so that (2.1) would not hold for a large time interval. This secularity is banished by choosing the constant of integration in t such that on the average over a 2π-interval G_0 disappears:

(2.8) $$\int_{\tau\nu}^{\tau\nu+2\pi} G_0(t,\tau)dt = 0$$

or

(2.9a) $$\frac{\partial C_0}{\partial\tau} = \frac{-1}{2\pi}\int_{\tau\nu}^{\tau\nu+2\pi} x_0(\bar{t},\tau)d\bar{t}.$$

Since at time $t_0+\pi$ the solution starts at the value $x = 2$, C_0 satisfies the initial value

(2.9b) $\quad C_0(0) = 2/3 - \alpha.$

The solution will leave the region A at a time $t_m = t_0 + 2\pi m$ as it approaches the line $x = 1$, which occurs when C_0 reaches the value $-2/3+\alpha$. From (2.9) it follows that in the slow time scale this will be for

$$(2.10) \qquad T(\alpha) = 2\pi \int_{2/3-\alpha}^{-2/3+\alpha} \left\{ \int_0^{2\pi} x_0(t;C_0)\,dt \right\}^{-1} dC_0.$$

Finally, the third equation reads

$$(2.11) \qquad 2\frac{\partial^2 x_0}{\partial t \partial \tau} + \frac{\partial^2 x_1}{\partial t^2} + (x_0^2-1)\left(\frac{\partial x_1}{\partial \tau} + \frac{\partial x_2}{\partial t}\right) + 2x_0 x_1 \left(\frac{\partial x_0}{\partial \tau} + \frac{\partial x_1}{\partial t}\right) +$$

$$+ (x_1^2 + 2x_0 x_2)\frac{\partial x_0}{\partial t} + x_1 = 0,$$

so according to (2.6) we have

$$(2.11) \qquad (x_0^2-1)x_2 = -\frac{\partial x_0}{\partial \tau} - \frac{\partial x_1}{\partial t} + \int_{t_0+\pi}^{t} \left\{ \int_{t_0+\pi}^{\bar{t}} \frac{\partial G_0}{\partial \tau}\, d\bar{\bar{t}} - G_1 \right\} d\bar{t} - x_1^2 x_0$$

with

$$(2.12) \qquad G_1 = x_1 + \partial C_1/\partial \tau.$$

The term $\partial G_0/\partial \tau$ satisfies the averaging condition and behaves as $a+b/(x_0^2-1)$, therefore its integral is not secular. The averaging condition

$$(2.13) \qquad \int_{\tau\nu}^{\tau\nu+2\pi} G_1(t,\tau)\,dt = 0,$$

yields a linear differential equation for C_1:

$$(2.14) \qquad \frac{\partial C_1}{\partial \tau} + \frac{C_1}{2\pi} \int_{\tau\nu}^{\tau\nu+2\pi} \frac{1}{x_0^2-1}\,dt = \frac{-\beta}{2\pi} \int_{\tau\nu}^{\tau\nu+2\pi} \frac{\sin t}{x_0^2-1}\,dt.$$

Let $C_1(0) = C_{10}$, then we have

$$C_1(\tau) = \exp\{\frac{-1}{2\pi}\int_0^{\tau}\int_{\bar{\tau}\nu}^{\bar{\tau}\nu+2\pi}\frac{1}{x_0^2-1}\,dtd\bar{\tau}\}$$

$$[C_{10} - \frac{\beta}{2\pi}\int_0^{\tau}\exp\{\frac{1}{2\pi}\int_0^{\bar{\tau}}\int_{\hat{\tau}\nu}^{\hat{\tau}\nu+2\pi}\frac{1}{x_0^2-1}\,dtd\hat{\tau}\}\int_{\bar{\tau}\nu}^{\bar{\tau}\nu+2\pi}\frac{\sin t}{x_0^2-1}\,dtd\bar{\tau}]$$

and so

(2.15a) $$C_1(T(\alpha)) = q(\alpha)\{C_{10} - \beta p(\alpha)\},$$

(2.15b) $$p(\alpha) = \frac{1}{2\pi}\int_0^{T(\alpha)}\exp\{\frac{1}{2\pi}\int_0^{\tau}\int_{\bar{\tau}\nu}^{\bar{\tau}\nu+2\pi}\frac{1}{x_0^2-1}\,dtd\bar{\tau}\}\int_{\tau\nu}^{\tau\nu+2\pi}\frac{\sin t}{x_0^2-1}\,dtd\tau,$$

(2.15c) $$q(\alpha) = \exp\{\frac{-1}{2\pi}\int_0^{T(\alpha)}\int_{\tau\nu}^{\tau\nu+2\pi}\frac{1}{\dot{x}_0^2-1}\,dtd\tau\}.$$

As t approaches the value $t_0 + T(\alpha)\nu$ each time for $t = t_m = t_0 + 2\pi m$ the solution gets closer to the line $x = 1$. In a $\nu^{-1/2}$-neighbourhood of $(x,t) = (1,t_m)$ x_0 would behave as

(2.16a) $$x_0 = 1 + \nu^{-1/2}v(t;\nu), \qquad v = \{\frac{1}{2}\alpha(t-t_m)^2\nu^{-1} + C_0'(T)(t_m-t_0-T\nu)\}^{1/2}$$

and x_1 as

(2.16b) $$x_1 = \frac{-\alpha(t-t_m)}{4v^2} - \frac{\beta+C_1(T)}{2v}, \qquad T = T(\alpha).$$

Clearly, the asymptotic expansion is not valid anymore, as $\nu^{-1}x_1$ increases in order of magnitude. Just before the solution enters such regions we have that asymptotically

$$v \approx -\frac{1}{2}\sqrt{2\alpha}(t-t_m)\nu^{-1/2} - C_0'(T)(t_m-t_0-\pi-T\nu)(t-t_m)^{-1}\nu^{1/2}/\sqrt{2\alpha},$$

and so

$$x \approx 1 + \nu^{-1/2}[-\tfrac{1}{2}\sqrt{2\alpha}(t-t_m)\nu^{-1/2} +$$

(2.17)

$$\{-\tfrac{1}{2}\sqrt{2\alpha} + \beta - C_1(T) - C_0'(T)(t_m - t_0 - \pi - T\nu)\}(t-t_m)^{-1}\nu^{1/2}/\sqrt{2\alpha}].$$

3. ASYMPTOTIC SOLUTION FOR REGION A_m

Let us assume for a moment that the solution has passed a $\nu^{-1/2}$ neighbourhood of a point $(x,t) = (1,t_{m-1})$ and returns to the interval $(1,2)$. Since the two variable expansion (2.1) is not valid anymore and the solution again approaches the line $x = 1$ at time $t = t_m$, we introduce for the time interval $t_{m-1} < t < t_m$ the expansion

$$x(t;\nu) = x_{m0}(t) + \nu^{-1}x_m(t) + \ldots . \tag{3.1}$$

Similar to (2.1) we obtain a recurrent system of differential equations for $x_{mi}(t)$, $i = 1,2,\ldots$. The first two equations read

$$(x_{m0}^2 - 1)\frac{dx_{m0}}{dt} = \alpha\cos t, \tag{3.2a}$$

$$(x_{m0}^2 - 1)\frac{dx_{m1}}{dt} + 2x_{m0}\frac{dx_{m0}}{dt}x_1 = -\frac{d^2x_{m0}}{dt^2} - x_{m0} + \beta\cos t. \tag{3.2b}$$

Integration yields

$$\frac{1}{3}x_{m0}^3 - x_{m0} = \alpha\sin t + C_0^{(m)} \tag{3.3a}$$

$$(x_{m0}^2 - 1)x_{m1} = -\frac{dx_{m0}}{dt} - \int_{t_{m-1}}^{t} x_{m0}(\bar{t})d\bar{t} + \beta\sin t + C_1^{(m)}. \tag{3.3b}$$

Since for $t = t_m$ x approaches the value 1, we have $C_0^{(m)} = \alpha - \frac{2}{3}$, so that

$$x_{m0}(t) = 2\cos\{\tfrac{1}{3}\arccos(\tfrac{3}{2}\alpha\sin t + \tfrac{3}{2}\alpha - 1)\}. \tag{3.4}$$

As t approaches t_m from below (3.1) behaves asymptotically as

$$x \approx 1 - \tfrac{1}{2}\sqrt{2\alpha}\,(t-t_m) + \nu^{-1}K_m/(t-t_m), \tag{3.5a}$$

(3.5b) $$K_m = -\frac{1}{2} + (-C_1^{(m)} + \beta + I_n)/\sqrt{2\alpha},$$

(3.5c) $$I_n = \int_{t_{m-1}}^{t_m} x_{m0}(t)\,dt.$$

Consequently, also the expansion (3.1) looses its validity as $t \uparrow t_m$.

4. ASYMPTOTIC SOLUTION FOR REGION B_m

We investigate the local behaviour of the solution in a $\nu^{-1/2}$-neighbourhood of $(x,t) = (1,t_m)$ by introducing the transformations

(4.1ab) $$t = t_m + \xi\nu^{-1/2}, \qquad x = 1 + v_m(\xi)\nu^{-1/2},$$

Substituting (4.1) into (1.1) and multiplying this equation with $\nu^{-1/2}$ we obtain, after taking the limit $\nu \to \infty$,

(4.2) $$\frac{d^2 v_{m0}}{d\xi^2} + 2\, v_{m0}\frac{dv_{m0}}{d\xi} = \alpha\xi.$$

The function $v_{m0}(\xi)$ expresses the local limit behaviour of the solution for $\nu \to \infty$. In order to match the solution of region A_m it must satisfy

(4.3) $$v_{m0}(\xi) = -\frac{1}{2}\,\xi\sqrt{2\alpha} + K_m/\xi, \qquad \xi \to -\infty,$$

see (3.5). The function

(4.4) $$v_{m0}(\xi) = -aD'_{K_m}(-a\xi)/D_{K_m}(-a\xi), \qquad a = \sqrt[4]{2\alpha},$$

satisfies (4.2) as well as (4.3). In (4.4) $D_\mu(z)$ denotes the so-called parabolic cylinder function of order μ (see WHITTAKER and WATSON [9, p. 347]) with

(4.5) $$D_\mu(z) = \exp(-\frac{1}{4}z^2)z^\mu\{1 - \frac{\mu(\mu-1)}{2z^2} + \ldots\}$$

for $z \to \infty$. Assuming that $K_m \geq 0$ the function $v_{m0}(\xi)$ will be regular for finite ξ, while for $\xi \to \infty$

(4.6) $$v_{m0}(\xi) \approx \frac{1}{2}\xi\sqrt{2\alpha} - (K_m+1)/\xi,$$

as

(4.7) $$D_\mu(z) = \exp(-\tfrac{1}{4}z^2)z^\mu\{1 - \frac{\mu(\mu-1)}{2z^2} + \ldots\}$$
$$- \frac{\sqrt{2\pi}}{\Gamma(-\mu)}\exp(\tfrac{1}{4}z^2+\mu\pi i)z^{-\mu-1}\{1 + \frac{(\mu+1)(\mu+2)}{2z^2} + \ldots\}$$

for $z \to -\infty$. On the other hand, at region A_{m+1}, the solution is approximated by

(4.8) $$x(t) \approx 1 + \frac{1}{2}\sqrt{2\alpha}(t-t_m) + \{-\frac{1}{2} + (C_1^{(m+1)}-\beta)/\sqrt{2\alpha}\}/(t-t_m)$$

as $t \uparrow t_m$. Thus, the local solution (4.4) of B_m matches the local solution of A_{m+1} if

(4.9) $$K_m + \frac{1}{2} = (\beta - C_1^{(m+1)})/\sqrt{2\alpha}.$$

Using (3.5b) we find that

(4.10) $$C_1^{(m+1)} = C_1^{(m)} - I_n.$$

Let for some m, say $m = n$,

(4.11) $$K_{n-1} \leq 0 < K_n \leq I_n/\sqrt{2\alpha}.$$

Then the parabolic cylinder function $D_{K_n}(-a\xi)$ vanishes for certain value(s) of the argument. Let $\xi = \xi_0$ be the lowest zero. For $\xi \uparrow \xi_0$ we have that

(4.12) $$v_{m0}(\xi) \approx (\xi-\xi_0)^{-1} + \frac{1}{3}a^2(\frac{1}{4}a^2\xi_0^2 - K_n - \frac{1}{2})(\xi-\xi_0),$$

so v_{m0} becomes singular and the solution rapidly decreases as ξ approaches ξ_0.

5. ASYMPTOTIC SOLUTION FOR REGION C

At this point the solution enters the boundary layer region C with local coordinate

(5.1) $$\eta = (t-t_n-\xi_0\nu^{-1/2})\nu.$$

Assuming that the solution can be expanded as

(5.2) $$x = W_0(\eta) + \nu^{-1}W_1(\eta) + \nu^{-2}W_2(\eta) + \dots,$$

we arrive at a recurrent system of equation for the coefficients W_i:

(5.3a) $$\frac{d^2W_0}{d\eta^2} + (W_0^2-1)\frac{dW_0}{d\eta} = 0,$$

(5.3b) $$\frac{d^2W_1}{d\eta^2} + (W_0^2-1)\frac{dW_1}{d\eta} + 2W_0W_1\frac{dW_0}{d\eta} = 0,\dots .$$

According to (4.12), (5.2) matches the solution for region B_n if for $\eta \to -\infty$

(5.4ab) $$W_0 \approx 1 + 1/\eta, \qquad W_1 \approx \frac{1}{3}a^2(\frac{1}{4}a^2\xi_0^2 - K_n - \frac{1}{2})\eta.$$

Condition (5.4a) is satisfied by the class of solutions

(5.5a) $$\frac{1}{1-W_0} + \frac{1}{3}\log\frac{W_0+2}{1-W_0} = -\eta + H_0,$$

while because of (5.4b) the integrated equation (5.3b) must have the form

(5.5b) $$\frac{dW_1}{d\eta} + (W_0^2-1)W_1 = a^2(\frac{1}{4}a^2\xi_0^2 - K_n - \frac{1}{2}).$$

As $\eta \to \infty$ the solution leaves the boundary layer region at exponential rate and for $t - t_n - \xi_0\nu^{-1/2}$ small, but independent of ν, (5.2) behaves asymptotically as

(5.6) $$x = -2 + \frac{1}{3}a^2(\frac{1}{4}a^2\xi_0^2 - K_n - \frac{1}{2})\nu^{-1} + O(\exp\{-3(t-t_n-\xi_0\nu^{-1/2})\nu\}).$$

6. CONDITIONS FOR $2\pi(2n-1)$-PERIODIC SOLUTIONS

As the solutions of period $T = 2\pi(2n-1)$, we are looking for, are symmetric, that is

$$(6.1)\qquad x(t) = -x(t-\tfrac{1}{2}T),$$

we have completed the local approximations. Transposing (5.6) to the complementary phase and substituting $t = t_0 + \pi + \xi_0\nu^{-1/2}$ in (2.1), we obtain the periodicity condition

$$(6.2)\qquad x_0(t_0+\pi+\xi_0\nu^{-1/2},\xi_0\nu^{-3/2}) + \nu^{-1}x_1(t_0+\pi+\xi_0\nu^{-1/2},\xi_0\nu^{-3/2})$$
$$= 2 - \tfrac{1}{3}a^2(\tfrac{1}{4}a^2\xi_0^2 - K_n - \tfrac{1}{2})\nu^{-1}$$

or

$$(6.3)\qquad K_n + \tfrac{1}{2} = (\beta+C_{10})/\sqrt{2\alpha}.$$

From (4.9) and (4.10) it follows that

$$(6.4)\qquad \sqrt{2\alpha}(K_n+\tfrac{1}{2}) = \beta - C_1^{(m)} + (n-m+1)I_n.$$

Comparing (2.17) with (3.5) we see that for $n-m \to \infty$, but at the rate such that $n-m = o(\nu)$, there exists a matching relation between $C_1(\tau)$ and $C_1^{(m)}$:

$$(6.5)\qquad C_1^{(m)} - I_n = C_1(T) + C_0'(T)(t_m-t_0-\pi-T\nu)$$

or, as $I_n = -2\pi C_0'(T)$,

$$(6.6)\qquad C_1^{(m)} = C_1(T) - (m-\tfrac{3}{2})I_n + (2\pi)^{-1}I_nT\nu.$$

Substituting (6.6) into (6.4), while using (2.15), we obtain

$$(6.7)\qquad \sqrt{2\alpha}(K_n+\tfrac{1}{2}) = (1-pq)\beta - qC_{10} + (n-\tfrac{1}{2})I_n - (2\pi)^{-1}I_nT\nu.$$

From (6.3) and (6.7) we derive

(6.8a) $\beta = \{\sqrt{2\alpha}(K_n + \frac{1}{2})(1+q(\alpha)) - G_n(\alpha)\}/S(\alpha)$,

(6.8b) $G_n(\alpha) = I_n(\alpha)\{(n - \frac{1}{2}) - (2\pi)^{-1}T(\alpha)\nu\}$

(6.8c) $S(\alpha) = 1 + q(\alpha) - p(\alpha)q(\alpha)$.

Since K_n may range from 0 to $I_n/\sqrt{2\alpha}$, a periodic solution of period $2\pi(2n-1)$ is possible for

(6.9) $$\{\frac{1}{2}\sqrt{2\alpha}(1+q(\alpha)) - G_n(\alpha)\}/S(\alpha) < \beta <$$

$$< \{(\frac{1}{2}\sqrt{2\alpha} + I_n)(1+q(\alpha)) - G_n(\alpha)\}/S(\alpha).$$

7. CONCLUDING REMARKS

The formal asymptotic analysis of the Van der Pol relaxation oscillator with a periodic forcing term as we presented in this report forms the last part of a series of studies on this problem, see [3, 4 and 5]. In [3] and [5] solutions of period $2\pi(2n-1)$ were found under given conditions for β for the special cases $\alpha = 0$ and $\alpha = 2/3$. If for $\alpha = 2/3$ we consider the range of β given by (6.9) we observe that this special case, studied in [5], is completely covered by the results of this report. As $I_n(2/3) = 6\sqrt{3}$, $T(2/3) = p(2/3) = 0$ and $q(2/3) = 0$, the conditions on $\beta_{2/3}$ read

(7.1) $$3\sqrt{3}(\frac{11}{18} - n) < \beta_{2/3} < 3\sqrt{3}(\frac{47}{18} - n).$$

For $\alpha = 0$ we have $I_n(0) = 2\pi$, $T(0) = 3/2 - \log 2$, $p(0) = 0$ and $q(0) = 1/2$, so

(7.2) $$(3 - 2\log 2)\nu - 2\pi(2n-1) < 3\beta_0 < (3 - 2\log 2)\nu - 2\pi(2n-2),$$

which for $\beta_0 \to \infty$ matches the conditions on this parameter given by inequalities (21) and (22) of [3].

It is our intention to write a final report in which we compare the asymptotic conditions on α, β and ν with numerical results on this problem, see FLAHERTY and HOPPENSTEADT [2]. For that purpose some integrals such as $I_n(\alpha)$ and $\mathcal{T}(\alpha)$ need to be evaluated by numerical integration or (if possible) by an analytical expression. Moreover, we will attempt to relate our formal asymptotic results with the outcome of topological-analytical work by GUCKENHEIMER [6] and LEVI [7].

REFERENCES

[1] COLE, J.D., *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.

[2] FLAHERTY, J.E. & F.C. HOPPENSTEADT, *Frequency entrainment of a forced Van der Pol oscillator*, Studies in Applied Math. 18 (1978), p. 5-15.

[3] GRASMAN, J., E.J.M. VELING & G.M. WILLEMS, *Relaxation oscillations governed by a Van der Pol equation with periodic forcing term*, SIAM J. Appl. Math. 31 (1976), p. 667-676.

[4] GRASMAN, J., M.J.W. JANSEN & E.J.M. VELING, *Asymptotic methods for relaxation oscillations*, in: *Proceedings of the Third Scheveningen Conference on Differential Equations*, W. Eckhaus and E.M. de Jager (eds.), North Holland Math. Studies 31 (1978), 93-111.

[5] GRASMAN, J., *Relaxation oscillations of a Van der Pol equation with large critical forcing term*, Quart. of Appl. Math. 38 (1980) p. 9-16.

[6] GUCKENHEIMER, J., *Symbolic dynamics and relaxation oscillations*, to appear in Physica D: Nonlinear phenomena.

[7] LEVI, M., *Periodically forced relaxation oscillations*, preprint.

[8] LITTLEWOOD, J.E., *On nonlinear differential equation of the second order*, III, Acta Math. 97 (1957), p. 267-308.

[9] WHITTAKER, E.T. & G.N. WATSON, *A course of modern analysis*, Cambridge, Cambridge University Press, (1935).